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Laplacian eigenvalues and fixed size multisection

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Abstract

For a simple and non-directed graph, bounds on a weighted bisection are related to min and max laplacian eigenvalues, respectively. The purpose of this article is to extend this result to the multisection case where each partition among k has fixed size; both bounds rely on eigenvalues of a certain Gram matrix together with k smallest and k greatest laplacian eigenvalues. These bounds are compared with known ones.
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Keywords: Max cut; Bisection; Multisection; Laplacian; Gram; Eigenvalues

1. Introduction

For a simple and non-directed graph, bounds on a weighted bisection are related to min and max laplacian eigenvalues, respectively. The purpose of this article is to extend this result to the multisection case where each partition among k has fixed size; both bounds rely on eigenvalues of a certain Gram matrix together with k smallest and k largest laplacian eigenvalues.

In Section 2, we recapitulate laplacian eigenvalues and bounds for bisection and notations. In Section 3, main result on bounds is derived from laplacian eigenvalues and eigenvalues of a certain Gram matrix that carries bisection case over the multisection one. In Section 4, computational results are given for a bunch of graphs together with a comparison with previously known bounds on them to show how new bounds compare favorably. In Section 5, results obtained so far with 0-eigenvalue property are enhanced to other eigenvalues, provided sibling eigenvector induces a simple eigensubspace structure.

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2. Laplacian eigenvalues and bisection

Given a simple non-directed graph $G=(V,E,W)$ with vertex set V, edge set E and labelling function $W:E\mapsto\mathbb{R}$ associating a weight to every edge, we now recapitulate the results known on partitioning of G into two parts. Let A be the adjacency matrix of G, i.e. for each edge $(i,j)\in E$, $A_{ij}=W_{(i,j)}$ (using subscripts as indexing functions for sake of conciseness). Then, laplacian matrix is defined by $L=\Delta-A$ where diagonal matrix Δ stores, for each vertex, the sum of weights of all adjacent edges $\Delta_i=\sum_{j\neq i}W_{(i,j)}$ (Δ is named after degree reference). From definition of Δ, in the case where all weights are positive reals, positive semi-definiteness of L follows which means that spectral structure entails strong meaning with respect to bisection.

Vectors are denoted by lowercase letters, matrices by uppercase letters. In particular, e refers to the vector all 1's and I to identity matrix. A bisection (equivalently a cocycle or a cut) of G through a set S of vertices is the set of weighted edges having one endpoint in S and the other in $V\backslash S$. Its characteristic vector x is defined as $x_i=1$ if $i\in S$, $x_i=-1$ otherwise and when no confusion arises the bisection is defined also by the sum of weights of associated edges $c=\sum_{i\in S}\sum_{j\in V\backslash S}W_{(i,j)}$.

The max-cut of G is the maximum value over all bisections. We borrow standard notations from linear algebra to relate laplacian matrix to bisection x through $4c=\langle Lx,x\rangle$ since only simple and undirected graphs are under consideration.

It is well known [2–4,9–11] that eigenvalues of L play a quite important role in bisection. We will refer to them in increasing order, $\lambda_1(L)\leqslant\lambda_2(L)\leqslant\cdots\leqslant\lambda_n(L)$. For sake of simplicity, we assume all multiplicities to be 1 to adjust maximum index to the size $n=|V|$. In the case of positive weights $\lambda_1(L)=0$. Among all eigenvectors, 0-eigenvector e (since $Le=0$ after laplacian definition) plays a crucial role. If we remove one occurrence of 0 in the multiset of eigenvalues, we refer to the other ones as $\lambda_1^*(L),\lambda_{n-1}^*(L)$.

Let a bisection with characteristic vector x be given by S with $s=|S|$ (without loss of generality, S is chosen such that $s\leqslant n/2$). Then $\|x\|^2=n$, $\langle e,x\rangle=(2s-n)\leqslant 0$.

A first bounding procedure comes after relaxation of $x_i=\pm1$ on any vertex $i\in V$:

Property 1 (Norm relaxation).

$$n\lambda_1(L)\leqslant 4c\leqslant n\lambda_n(L).$$

Proof. Let us consider an orthogonal basis of eigenvectors $\{e_i \mid \|e_i\|^2=n,\ i=1,\dots,n\}$ where squared norms ($\|e_i\|^2=n$) fit bisection requirement (like $e=e_m$). Then any bisection can be written as a linear combination: $x=\sum\alpha_i e_i=\alpha_m e+y$. Using orthogonality of the basis, $\|x\|^2=\sum\alpha_i^2\|e_i\|^2$ follows, so that $\sum\alpha_i^2=1$ is implied.

In the same way, $4c=\langle Lx,x\rangle=\sum\alpha_i^2\lambda_i(L)\|e_i\|^2$ and hence we have the following bounds:

$$n\lambda_1(L)\leqslant\langle Lx,x\rangle\leqslant n\lambda_n(L).\qquad\square$$

The lower bound has no interest in the case of positive weights, since the bisection is then obviously positive and $\lambda_1=0$.

A tighter bounding procedure involves the fact that e is an eigenvector for the eigenvalue 0:

Property 2 (Single 0-eigenvector relaxation).

$$\frac{s(n-s)}{n}\lambda_1^*(L) \leqslant c \leqslant \frac{s(n-s)}{n}\lambda_{n-1}^*(L).$$

Proof. We use Property 1 and remove the (null) contribution of $e = e_m$ in the sum $\sum \alpha_i^2 \lambda_i(L) \|e_i\|^2$. Thus

$$n(1-\alpha_m^2)\lambda_1^*(L) \leqslant \langle Lx, x\rangle = 4c \leqslant n(1-\alpha_m^2)\lambda_{n-1}^*(L)$$

and $\alpha_m = \langle x, e\rangle / \langle e, e\rangle = (2s-n)/n$ gives $n(1-\alpha_m^2) = (4s(n-s))/n$. $\square$

This results appears in [10] as Lemma 3.1, under the (in fact not necessary) restriction of positive weights.

Since $Le = 0$, the expression $4c = \langle Lx, x\rangle$ is the trace $\mathrm{tr}(Q)$ of the matrix

$$Q = \begin{bmatrix} \langle Lx, x\rangle & \langle Lx, e\rangle \\ \langle Le, x\rangle & \langle Le, e\rangle \end{bmatrix}.$$

Then a $(2,2)$-matrix B such that $(x, e)B = (x', e')$ gives

$$B^{\mathrm{t}}QB = Q' = \begin{bmatrix} \langle Lx', x'\rangle & \langle Lx', e'\rangle \\ \langle Le', x'\rangle & \langle Le', e'\rangle \end{bmatrix},$$

and thus $\mathrm{tr}(Q') = \mathrm{tr}(QBB^{\mathrm{t}})$ is easily computed when BB^{t} or Q is scalar.

Hence

Property 3 (Double 0-eigenvector relaxation).

$$2(n-s)\lambda_1(L) + 2s\lambda_2(L) \leqslant 4c \leqslant 2s\lambda_{n-2}(L) + 2(n-s)\lambda_{n-1}(L).$$

Proof. We replace the set of vectors $\{x, e\}$ by the set of orthogonal vectors $\{(e+x)/2, (e-x)/2\}$ that generate the same space and whose supports are S and $V \backslash S$ (these vectors were indeed used in the original proof by Donath and Hoffman). Note that

$$B = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

which describes the change of vectors satisfies $BB^{\mathrm{t}} = \frac{1}{2}I$. Moreover the squared lengths of these two vectors are s and $n-s$.

The contribution to the trace of a pair of orthogonal vectors of length 1 is between $\lambda_1(L) + \lambda_2(L)$ and $\lambda_{n-1}(L) + \lambda_n(L)$, the sums of the smallest and largest eigenvalues, and the contribution of a single orthogonal vector of length 1 is between $\lambda_1(L)$ and $\lambda_n(L)$, according to Rayleigh principle. $\square$

Besides, we may note that $2s \leqslant 2n-2s$ are the two eigenvalues of the Gram matrix of the two vectors, e and x, i.e.

$$\begin{bmatrix} \langle x,x\rangle & \langle x,e\rangle \\ \langle e,x\rangle & \langle e,e\rangle \end{bmatrix}.$$

We may also use the set $\{y,e\}$ of orthogonal vectors, where $y = x - [\langle e,x\rangle/n]e$ has squared length $4s(n-s)/n$.

Moreover, Donath and Hoffman notice that for any diagonal matrix D with null trace, $\langle Dx,x\rangle = \mathrm{tr}(D) = 0$ holds. Therefore, the previous bounds could be tightened:

Property 4 (Null trace relaxations).

$$n \max_{D,\mathrm{tr}(D)=0} \lambda_1(L+D) \leqslant 4c \leqslant n \min_{D,\mathrm{tr}(D)=0} \lambda_n(L+D),$$

$$\max_{D,\mathrm{tr}(D)=0} ((n-s)\lambda_1(L+D) + s\lambda_2(L+D)) \leqslant 2c,$$

$$2c \leqslant \min_{D,\mathrm{tr}(D)=0} ((n-s)\lambda_n(L+D) + s\lambda_{n-1}(L+D)).$$

Finally, the mapping $D \mapsto -D - \Delta + [\mathrm{tr}(\Delta)/n]I$ is clearly one-to-one and onto on the set of diagonal matrices having null trace $\mathrm{tr}(D) = 0$. Therefore, we have the following property.

Property 5 (Laplacian vs. adjacency eigenvalues).

$$\min_{D,\mathrm{tr}(D)=0} \lambda_n(L+D) + \max_{D',\mathrm{tr}(D')=0} \lambda_1(A+D') = \frac{\mathrm{tr}(\Delta)}{n},$$

$$\max_{D,\mathrm{tr}(D)=0} \lambda_1(L+D) + \min_{D',\mathrm{tr}(D')=0} \lambda_n(A+D') = \frac{\mathrm{tr}(\Delta)}{n}$$

which relates the highest and the lowest optimized eigenvalues of laplacian and adjacency matrix, respectively. Note that $\mathrm{tr}(\Delta)$ is just twice the sum of the weights of all edges.

3. Laplacian eigenvalues and multisection

Previous section carries over multisection of graph G, provided each part size is fixed, namely $n_1,\ldots,n_k$ for a k-partition. A multisection of G through a set P of fixed size parts is the set of weighted edges having one endpoint in a part from P and the other outside; the characteristic vector associated with any part p is written as $\mathbf{x}^p$ where

$$\mathbf{x}_i = \begin{cases} 1 & \text{if } i \in p, \\ -1 & \text{otherwise.} \end{cases}$$

The value of the multisection (also called multisection for brevity) is defined by the sum of weights of edges between its parts; in other words, it is given by $8\mathbf{c}_P = \sum_{p \in P} \langle L\mathbf{x}^p, \mathbf{x}^p \rangle$ where as before L is laplacian matrix and $n = \sum_{j=1}^{k} n_j$ is the number of vertices in G.

Property 6 (Poor norm relaxation).

$$nk\lambda_1(L) \leqslant 8\mathbf{c}_P \leqslant nk\lambda_n(L).$$

Proof. From bisection, we get that given any part p_j

$$n\lambda_1(L) \leqslant 4c_{p_j} = \langle L\mathbf{x}^j, \mathbf{x}^j \rangle \leqslant n\lambda_n(L)$$

gives weighted cut c_{p_j} between part p_j and $V \setminus p_j$. Summing over all parts p_j, $j = 1, \ldots, k$ will count weighted cut for given multisection twice. Therefore

$$nk\lambda_1(L) \leqslant 8\mathbf{c}_P = \sum_{j=1}^{k} \langle L\mathbf{x}^j, \mathbf{x}^j \rangle = \mathrm{tr}(\langle LP, P \rangle) \leqslant nk\lambda_n(L),$$

where for sake of conciseness, we confuse the set P with its columnwise representation as $[\mathbf{x}^1, \ldots, \mathbf{x}^j, \ldots, \mathbf{x}^k]$. $\square$

To improve this poor bounding, we can use the multiple 0-eigenvector relaxation instead. Let us introduce, as in bisection case, $\mathbf{y}^i = \mathbf{x}^i - \langle \mathbf{x}^i, e \rangle / n$ where $\mathbf{x}^i$ is the characteristic vector associated with part i. Then, clearly $\langle e, \mathbf{y}^i \rangle = 0$.

Thus $\mathbf{y}^i$ is orthogonal to e and its length is $4s^i(n - s^i)/n$. Moreover, $\langle \mathbf{y}^i, \mathbf{y}^j \rangle = -4s^i s^j / n$, where s^i is the number of vertices in the part p^i of P.

Therefore, we introduce further Gram matrix Γ whose entries are $\Gamma_{ij} = \langle \mathbf{y}^i, \mathbf{y}^j \rangle$ to refine multisection case, thanks to the following lemma:

Lemma 1. *Let E be an euclidian vector space of dimension n, Q an auto-adjoint operator on E, $\{V_i,\ i = 1, \ldots, k\}$ a family of vectors of E, with Gram matrix Γ, i.e. $\Gamma_{ij} = \langle V_i, V_j \rangle$. Then*

$$\sum_{i=1}^{k} \gamma_i \lambda_{k+1-i} \leqslant \sum_{i=1}^{k} \langle QV_i, V_i \rangle \leqslant \sum_{i=1}^{k} \gamma_i \lambda_{n-k+i},$$

where the λ_i's (resp. γ_i's) are the Q-eigenvalues (resp. Γ-eigenvalues) in increasing order.

Proof. Let W be the matrix whose columns are coordinates of $\{V_i \mid i = 1, \ldots, k\}$ family in any orthogonal basis of E. Without loss of generality, we may choose a basis such that matrix associated with Q is diagonal, say $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then, $\sum_{i=1..k} \langle QV_i, V_i \rangle = \mathrm{tr}(\langle DW, W \rangle)$. Since $\Gamma = \langle W, W \rangle$ is positive semi-definite ($\langle \Gamma x, x \rangle = \langle \langle W, W \rangle x, x \rangle = \|Wx\|^2 \geqslant 0$), there exists an orthogonal $k \times k$ matrix P such that $\langle WP, WP \rangle = \mathrm{diag}(\gamma_1, \ldots, \gamma_k)$. Using $\mathrm{tr}(\langle DW, W \rangle) = \mathrm{tr}(\langle DWP, WP \rangle)$, we are done. The end of the proof is essentially the one by Donath and Hoffman's [4]. $\square$

For fixed size multisection, we now directly have

Property 7 (Single 0-eigenvector relaxation).

$$\sum_{i=1}^{k} \gamma_i \lambda^*_{k+1-i}(L) \leqslant 8\mathbf{c}_P \leqslant \sum_{i=1}^{k} \gamma_i \lambda^*_{n-1-k+i}(L).$$

We apply the preceding formula to the restriction of the operator L to the stable space orthogonal to e. Since $\gamma_1=0$, only the $k-1$ smallest and $k-1$ largest eigenvalues of this restriction are used.

Remark 1. This kind of result dates back to Hardy, Littlewood and Pólya [6] who worked on a special symmetric bilinear form and whose result was recently refined (1994) by Çela and Woeginger [1]. Some developements are also described in [7].

As in bisection case, let $\mathbf{y}^i = \mathbf{x}^i - [\langle e, \mathbf{x}^i\rangle/n]e$. Then $\mathbf{y}^i$ is orthogonal to e and its length is $4s(n-s)/n$. Moreover, if $i \neq j$, then $\langle \mathbf{y}^i \mathbf{y}^j\rangle = -4s^i s^j/n$, where s^i is the number of vertices in the part p^i of P, since $\langle e + \mathbf{x}^i, e + \mathbf{x}^j\rangle = 0$, owing to $p^i \cap p^j = \emptyset$.

The maximum multisection c for k parts satisfies $4c \leqslant n\sum_{i=n-k+1}^{n-1} \lambda_i(L)$. This upper bound comes from a partition into k parts of sizes n/k. This value is sometimes better and sometimes worse than the value $n[(k-1)/2k]\lambda_n(L+D)$ that appears (after carrying duality) in [5] or [8].

4. Computational results

4.1. Lower bounds for cube, torus and mesh

The b-ary hypercube $Q = K_b \times K_b \times \cdots \times K_b$ of dimension n has laplacian eigenvalues $0, b, 2b, \dots, nb$ with multiplicities $1, n(b-1), \dots, \binom{n}{k}(b-1)^k, \dots, (b-1)^n$ (summing to $b^n = N$ of course).

Thus, separating Q in 2 parts of sizes $m, N-m$ cuts at least $bN(N-m)/N$ edges, that is $bN(1-\Phi)$ with $\Phi = m/N$.

The cycle C_b has laplacian eigenvalues $2 - 2\cos 2k\pi/b$, $k = 0..b-1$. Hence the small eigenvalues of the torus $C_b \times C_b$ are 0 (multiplicity 1) and then $2 - 2\cos 2\pi/b$ (multiplicity 4).

Hence the separation into 2 parts of sizes $m, b^2 - m$ has the lower bound $(2 - 2\cos 2\pi/b)m(b^2-m)/b^2$. This bound is very poor since $2 - 2\cos 2\pi/b$ is approximately $4\pi^2/b^2$ when b is large.

The path P_b has laplacian eigenvalues $2 - 2\cos k\pi/b$, $k = 0..b-1$. Hence, the small eigenvalues of the grid $P_b \times P_b$ are 0 (multiplicity 1) and then $2 - 2\cos \pi/b$ (multiplicity 2).

Hence the separation into 2 parts of sizes $m, b^2 - m$ has the lower bound $(2 - 2\cos \pi/b)m(b^2-m)/b^2$, again a very poor bound. In Table 1, we compare Donath and Hoffman's bound with the bound introduced in previous section.

Table 1
Lower bounding

Graph	Parts	Don. & Hoff.	Delorme	# cuts	Laplacian eigenvalues$^{\text{multiplicity}}$	Gram eigenvalues$^{\text{multiplicity}}$
3-Cube	$2,3,3$	5	$\frac{21}{4}$	7	$0,2^3,4^3,6$	$0,9,12$
4-Cube	$12,2,2$	4	$\frac{13}{2}$	10	$0,2^4,4^6,6^4,8$	$0,8,18$
K_3^3					$0,3^6,6^{12},9^8$	
	$9,9,9$	27	27	27		$0,36^2$
	$3,9,15$	18	23	24		$0, \frac{92}{3} - \frac{-8\sqrt{31}}{3}, \frac{92}{3} + \frac{8\sqrt{31}}{3}$
$C_4 \times C_3$					$0,2^2,3^2,4,5^4,7^2$	
	$2,3,3,4$	9	$9.93 \simeq \frac{247-\sqrt{73}}{24}$	13		$0, \frac{35-\sqrt{73}}{3}, 12, \frac{\sqrt{35+\sqrt{73}}}{3}$
	$3,4,5$	7	$7.83 \simeq \frac{47}{6}$	9		$0, \frac{40}{3}, 18$
P_3					$0,1,3$	
	$1,2$	$2-\sqrt{2} \simeq 0.58$	$\frac{2}{3}$	1		$0, \frac{16}{3}$
P_6					$0,2-\sqrt{3},1,2,3,2+\sqrt{3}$	
	$1,5$	0.14	$0.22 \simeq \frac{5}{6}(2-\sqrt{3})$	1		$0, \frac{20}{3}$
	$2,4$	0.31	$0.35 \simeq \frac{8}{6}(2-\sqrt{3})$	1		$0, \frac{32}{3}$
	$3,3$	0.55	$0.40 \simeq \frac{3}{2}(2-\sqrt{3})$	1		$0,12$
	$2,2,2$	$3-\sqrt{3} \simeq 1.27$	$1.27 \simeq 3-\sqrt{3}$	2		$0,8^2$
	$1,1,4$	$\frac{3-\sqrt{3}}{2} \simeq 0.63$	$0.77 \simeq \frac{5}{2}-\sqrt{3}$	2		$0,4,8$
$P_3 \times P_3$					$0,1^2,2,3^2,4^2,6$	
	$2,3,4$	2.86	$2.88 \simeq \frac{26}{9}$	5		$0, \frac{104-8\sqrt{7}}{9}, \frac{104+8\sqrt{7}}{9}$
$P_4 \times P_4$					$0,(2-\sqrt{2})^2,4-2\sqrt{2},4-\sqrt{2},$ $(2+\sqrt{2})^2,4^3,(4+\sqrt{2})^2,4+2\sqrt{2}$	
	$5,5,6$	3.44	$3.11 \simeq (2-\sqrt{2})\frac{85}{8}$	8		$0,20,\frac{45}{2}$
	$2,7,7$	3.06	2.81	7		$0,\frac{21}{2},28$
	$2,2,12$	1.24	1.901	5		$0,8,18$
Coxeter					$0,1^8,(4-\sqrt{2})^6,4^7,(4+\sqrt{2})^6$	
	$6,11,11$	$\frac{17}{2} \simeq 8.5$	9.03	13		$0,\frac{198}{7},44$
$\mathbb{Z}_{13},\pm 1,\pm 5$					$0,2.62^4,3.72^4,6.65^4$	
	$4,4,5$	10.42	11.2	13		$0,16,\frac{240}{13}$

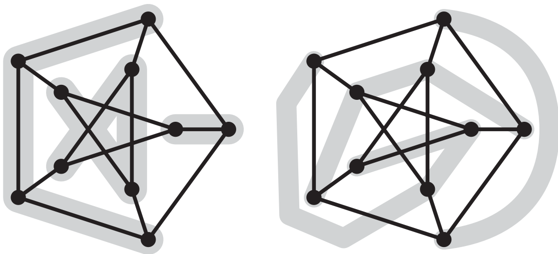

Fig. 1. Petersen graph.

It should be noted that the Gram matrix of vectors $\mathbf{y}^i$ has eigenvalue 0 since $\sum \mathbf{x}^i = (k-2)e$ and thus $\sum \mathbf{y}^i = 0$. Moreover, if t parts have cardinality $s \geqslant 2$, then $4s$ is an eigenvalue of the Gram matrix, with multiplicity at least $s-1$, with eigenvectors $\mathbf{y}^i - \mathbf{y}^j$ where p^i and p^j have s elements.

4.2. Petersen graph

We want to split it into parts of sizes $2, 4, 4$. As shown in Fig. 1, we can achieve 14 edges, that is the maximum and 8, that is the minimum. This can be proven from the following two facts:

- Petersen graph has girth 5 (hence at most 3 edges can lie inside a part with 4 vertices).
- Stable sets can have 4 vertices, but any two such stables sets on 4 vertices meet (hence it is not possible to have all 15 edges outside the parts).

But the game here is to rely on eigenvalues and symmetry only.

Here the laplacian eigenvalues and multiplicities are $0, 2^5, 5^4$. The trace of the Gram matrix is 25.6. The eigenvalue 2 gives $c \geqslant 6.4$ (Donath and Hoffman technique gives here only 6) and the eigenvalue 5 gives $c \leqslant 16$.

5. Using other eigenvalues

Poor bounds have been achieved in property 7 whenever Gram matrix is based on 0-eigenvector only. In order to improve the bounds, we are looking for a small eigensubspace, say e along with another eigenvector, having a simple structure, to be able to compute a relatively small number of dedicated contributions associated with different partitions on both eigensubspace and its orthogonal complement. In this section, improvements over known bounds are reported on antiprism and Petersen graphs. Provided a tractable structure for both eigensubspace and its orthogonal complement, this technique settles foundations for hand computed eigenspace splitting in possibly higher dimension.

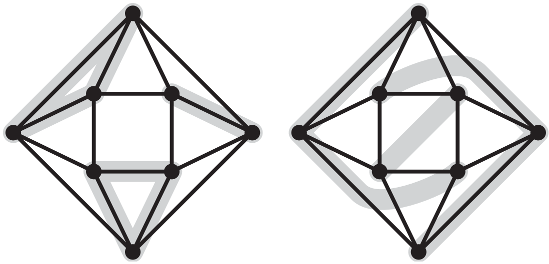

Fig. 2. The antiprism.

Table 2
Results for the antiprism

1's common to				
K and $\mathbf{x}^i$	$3,0,1$	$3,1,0$	$2,1,1$	$2,2,0$
$\mathbf{y}^i \cdot K$	$6,-6,0$	$6,-2,-4$	$4,2,0$	4,4,0
$[\mathbf{z}^i \cdot \mathbf{z}^j]$	$\begin{bmatrix} 3 & 0 & -3 \\ 0 & 3 & -3 \\ -3 & -3 & 6 \end{bmatrix}$	$\begin{bmatrix} 3 & -3 & 0 \\ -3 & 7 & -4 \\ 0 & -4 & 4 \end{bmatrix}$	$\begin{bmatrix} 7 & -4 & -3 \\ -4 & 7 & -3 \\ -3 & -3 & 6 \end{bmatrix}$	$\begin{bmatrix} 7 & -5 & -2 \\ -5 & 7 & -2 \\ -2 & -2 & 4 \end{bmatrix}$
Eigenvalues	$0,3,9$	$0,7 \pm \sqrt{13}$	$0,9,11$	$0,6,12$
Contribution	9/8	7/8	1/8	3/8
Lower bound	10.62	9.77	7.21	8.06
Upper bound	14.96	14.72	14.28	14.43

5.1. The antiprism

We consider the graph of Fig. 2, and we partition its vertices into parts of sizes $3,3,2$. Its laplacian eigenvalues and multiplicities are $0,(4-\sqrt{2})^2,4,(4+\sqrt{2})^2,6^2$. The eigenvalues of the Gram matrix

$$\begin{bmatrix} 15/2 & -9/2 & -3 \\ -9/2 & 15/2 & -3 \\ -3 & -3 & 6 \end{bmatrix}$$

of the $\mathbf{y}^i$'s are $0,9,12$, its trace 21.

We have already the bounds $\frac{1}{8}(6 \cdot 12+(4+\sqrt{2}) \cdot 9)1=\frac{63}{4}=15.09$ and $\frac{1}{8}(4-\sqrt{2}) \cdot 21=6.78$.

The eigenvalue 6 has an eigenvector K of squared length 8, made from four 1's and four -1's.

This allows up to symmetry four cases detailed in Table 2, where $\mathbf{z}^i=\mathbf{y}^i-[\mathbf{y}^i \cdot K/8]K$, and the contribution (to eigenspace associated to eigenvalue 6) is $\sum(\mathbf{y}^i \cdot K)^2/64$. The lower bound and upper bounds are $(4 \pm \sqrt{2})\sum\|\mathbf{z}^i\|^2+6\sum(\mathbf{y}^i \cdot K)^2/64$.

Table 3
Bounds obtained for Petersen graph

$[4,4,2]$ u.b.		$[4,4,2]$ l.b.	
$[4,0,1]$	11.2	$[4,0,0]$	12
$[4,1,0]$	12.4	$[3,0,1]$	9.25
$[3,1,1]$	14.8	$[2,0,2]$	9
$[3,2,0]$	14.8	$[3,1,0]$	8.25
$[2,2,1]$	16	$[1,1,2]$	7.75
		$[2,2,0]$	7
		$[2,1,1]$	6.75

Thus at least 8 and at most 14 edges join the parts. (Actual values 9 and 14 are shown in Fig. 2.)

5.2. Petersen graph again

There exists an eigenvector K for eigenvalue 2 with five 1's and five -1's. Indeed, there exist 12 such eigenvectors that span the eigenspace for eigenvalue 2. Up to symmetry, there are only 5 cases to consider for the number of 1 coordinates in the 3 parts. Using K and the all 1's vector e, one improves upper bounds.

Similarly, we may use an eigenvector for eigenvalue 5 with four 3's and six -2's and discuss the number of 3's in the parts to improve lower bounds. The results are recorded in Table 3.

Now, the only way to get 15 edges (or more) between the parts would be, according to the first array, to manage to have the three characteristic vectors of the parts orthogonal to all these eigenvectors for eigenvalue 2 made from 1 and -1's. Since the 12 such eigenvectors span the eigenspace, all three characteristic vectors should be orthogonal to that eigenspace. Then there would be 16 edges between the parts, because the characteristic vectors would belong to the sum of the eigenspaces of eigenvalues 0 and 5. Since the graph has only 15 edges, this is impossible. Thus we have proven with eigenvalues and eigenvectors that the maximum is 14.

6. Concluding remarks

In this article, we described how simple computations on laplacian eigenvalues of a simple non-directed graph and eigenvalues of a certain Gram matrix provide bounds for a weighted multisection under fixed size partitioning assumption.

By carefully watching eigenvector structure, those bounds are refined to some extent. Another direction of possible improvements would be to factorize somehow laplacial spectrum through graph automorphism in order to push pull bounds by the null trace trick, i.e. to use invariance of the bounds under addition of a diagonal matrix with null trace.

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